The structure of polynomial operations associated with smooth digraphs

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Digraphs, homomorphisms and polymorphisms

Definition

A **digraph** is a pair $\mathbb{G} = (G; \to)$, where G is the set of **vertices** and $\to \subseteq G^2$ is the set of **edges**.

Definition

A **homomorphism** from $\mathbb{G}$ to $\mathbb{H}$ is a map $f : G \to H$ that preserves edges:

$$a \to b \text{ in } \mathbb{G} \quad \Longrightarrow \quad f(a) \to f(b) \text{ in } \mathbb{H}.$$

$\mathrm{Hom}(\mathbb{G}, \mathbb{H}) = \{ f \mid f : \mathbb{G} \to \mathbb{H} \}$, write $\mathbb{G} \to \mathbb{H}$ iff $\mathrm{Hom}(\mathbb{G}, \mathbb{H}) \neq \emptyset$.

Definition

The clone of **polymorphisms** of $\mathbb{G}$ is $\mathrm{Hom}(\mathbb{G}) = \bigcup_{n=1}^{\infty} \mathrm{Hom}(\mathbb{G}^n, \mathbb{G})$.

CSP and cores

Definition

The **constraint satisfaction problem** for template $\mathbb{H}$ is the membership problem for

$$\mathrm{CSP}(\mathbb{G}) = \{\, \mathbb{H} \mid \mathbb{H} \to \mathbb{G} \,\}.$$

Proposition

$\to$ is a quasi-order on the set of finite digraphs. If $\mathbb{G}$ is a minimal member of the $\leftrightarrow$ class of $\mathbb{H}$, then

- *every endomorphism of $\mathbb{G}$ is an automorphism,*
- *$\mathbb{G}$ is uniquely determined up to isomorphism, and*
- *$\mathbb{G}$ is isomorphic to an induced substructure of $\mathbb{H}$.*

Definition

$\mathbb{G}$ is a **core** if it has no proper endomorphism. The **core of** $\mathbb{H}$ is the uniquely determined core structure in the $\leftrightarrow$ class of $\mathbb{H}$.

Finite duality and exponentiation

- set of finite relational structures modulo $\leftrightarrow$ is a partially ordered set
- isomorphic to the set of core isomorphism types
- minimal [maximal] element: 1-element structure, with empty [full] relations
- join: disjoint union, meet: direct product,
- satisfies distributive laws, join irreducible = connected
- Heyting algebra (relatively pseudocomplemented)
- $\mathbb{F} \wedge \mathbb{G} \to \mathbb{H} \iff \mathbb{H}^{\mathbb{F}\times\mathbb{G}} = \left(\mathbb{H}^{\mathbb{G}}\right)^{\mathbb{F}}$ has a loop $\iff \mathbb{F} \to \mathbb{H}^{\mathbb{G}}$
- if $\mathbb{G}$ is join irreducible with lower cover $\mathbb{H}$, then $(\mathbb{G}, \mathbb{H}^{\mathbb{G}})$ is a dual pair

Theorem (Nešetřil, Tardif, 2010)

Let $\mathbb{G}$ be a finite connected core structure. Then $\mathbb{G}$ has a dual pair $\mathbb{H}$, i.e. $\{\,\mathbb{F} \mid \mathbb{F} \to \mathbb{G}\,\} = \{\,\mathbb{F} \mid \mathbb{H} \not\to \mathbb{F}\,\}$, if and only if $\mathbb{G}$ is a tree.

Exponentiation

Definition

Let $\mathbb{H}^{\mathbb{G}}$ be the digraph on the set H^G with edge relation $f \to g$ iff

$$a \to b \text{ in } \mathbb{G} \implies f(a) \to g(b) \text{ in } \mathbb{H}.$$

Proposition

- $\mathrm{Hom}(\mathbb{G}, \mathbb{H}) = \{f \in \mathbb{H}^{\mathbb{G}} : f \to f\}$
- $\mathbb{G}^n = \mathbb{G}^{\mathbb{L}_n}$ *where* $\mathbb{L}_n = (\{1, \dots, n\}; =)$
- $\left(\mathbb{H}^{\mathbb{G}}\right)^{\mathbb{F}} = \mathbb{H}^{\mathbb{G} \times \mathbb{F}}$
- $\mathbb{H}^{\mathbb{F}} \times \mathbb{G}^{\mathbb{F}} = (\mathbb{H} \times \mathbb{G})^{\mathbb{F}}$
- *the composition map* $\circ : \mathbb{H}^{\mathbb{G}} \times \mathbb{G}^{\mathbb{F}} \to \mathbb{H}^{\mathbb{F}}$ *is a homomorphism*
- *If* $f \to g$ *in* $\mathbb{H}^{\mathbb{G}^n}$ *and* $f_1 \to g_1, \dots, f_n \to g_n$ *in* $\mathbb{G}^{\mathbb{F}}$, *then*

$$f(f_1, \dots, f_n) \to g(g_1, \dots g_n) \text{ in } \mathbb{H}^{\mathbb{F}}$$

$\mathbb{G}^{\mathbb{G}}$ IS INTERESTING

- $\mathbb{G}^{\mathbb{G}}$ has a loop at id, so every instance of CSP($\mathbb{G}^{\mathbb{G}}$) has a solution. Can we test for non-trivial solutions?
- $\left(\mathbb{G}^{\mathbb{G}}\right)^{\mathbb{H}} = \mathbb{G}^{(\mathbb{G}\times\mathbb{H})}$
- If $\mathbb{G}$ is a core and we can solve CSP($\mathbb{G}$), then we can test if an instance of CSP($\mathbb{G}^{\mathbb{G}}$) has a non-trivial solution.
- If we can test for non-trivial solutions in CSP($\mathbb{G}^{\mathbb{G}}$), then we can solve CSP($\mathbb{G}$).
- Connectivity properties of $\mathbb{G}^{\mathbb{G}}$ sometimes can be lifted to the set of solutions in $\left(\mathbb{G}^{\mathbb{G}}\right)^{\mathbb{H}}$.

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Connectivity in $\mathbb{G}^{\mathbb{G}}$

Theorem (Gyenizse; 2013)

Suppose, that $|\mathbb{G}| \geq 6$. *Then* $\mathbb{G}^{\mathbb{G}}$ *is connected if and only if*

- $\mathbb{G}$ *is empty,*
- *there exists* $a \in G$ *such that* $a \to x$ *for all* $x \in G$*, or*
- *there exists* $a \in G$ *such that* $x \to a$ *for all* $x \in G$.

Definition

$\mathrm{End}(\mathbb{G})$ is the induced subgraphs of $\mathbb{G}^{\mathbb{G}}$ on $\mathrm{Hom}(\mathbb{G}, \mathbb{G})$.
$\mathrm{Aut}(\mathbb{G}) = \mathrm{End}(\mathbb{G}) \cap \mathrm{Sym}(G)$.

Theorem (Gyenizse; 2013)

$\mathrm{Aut}(\mathbb{G})$ *is a disjoint union of complete digraphs. The number of elements in each component is the same and is a product of factorials.*

Structure, polymorphisms and connectivity

Theorem (Larose, Zádori; 1997)

If $\mathbb{G}$ is a connected poset and has Maltsev polymorphisms, then $\mathrm{End}(\mathbb{G})$ *is connected.*

Theorem (Larose, Loten, Zádori; 2005)

If $\mathbb{G}$ is connected, reflexive, symmetric and has Hobby-McKenzie polymorphisms (for omitting types $\mathbf{1}$ *and* $\mathbf{5}$*), then* $\mathrm{End}(\mathbb{G})$ *is connected.*

Theorem (M, Zádori; 2012)

If $\mathbb{G}$ is connected, reflexive and has Hobby-McKenzie polymorphisms, then $\mathrm{End}(\mathbb{G})$ *is connected.*

Collapse of Maltsev conditions

Theorem (Larose, Loten, Zádori; 2005)

If a finite reflexive and symmetric digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism

Theorem (M, Zádori; 2012)

If a finite reflexive digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism and totally symmetric polymorphisms for all arities.

Theorem (Kazda; 2011)

If a finite digraph has a Maltsev polymorphism, then it has a majority polymorphism.

How far can we push this?

- We need a structural property on $\mathbb{G}$
- We need an induced subgraph of $\mathbb{G}^{\mathbb{G}}$
- We need some polymorphisms of $\mathbb{G}$

Reduction of CSP to digraphs

Theorem (Bulín, Delić, Jackson, Niven; 2013)

For every finite relational structure $\mathbb{A}$ *there exists a finite digraph* $\mathbb{G}$*, such that* CSP($\mathbb{A}$) *and* CSP($\mathbb{G}$) *are polynomially equivalent and almost all Maltsev conditions, e.g.*

- *Taylor term,*
- *Willard terms,*
- *Hobby-McKenzie terms,*
- *Gumm terms,*
- *edge term,*
- *Jónsson terms,*
- *near-unanimity term,*
- *but not Maltsev term*

hold equivalently by $\mathbb{A}$ *and* $\mathbb{G}$*.*

Definition

The digraph $\mathbb{G}$ is **smooth** if its edge relation is subdirect (no sources and sinks).

Definition

The **algebraic length** of a directed path is the number of forward edges minus the number of backward edges. The algebraic length of $\mathbb{G}$ is the smallest positive algebraic length of oriented cycles (closed paths) of $\mathbb{G}$.

- If $\mathbb{G}^2$ is connected, then $\mathbb{G}$ is connected, has algebraic length 1 and has no source or no sink.
- If $\mathbb{G}$ is smooth, algebraic length 1 and (strongly) connected, then $\mathbb{G}^n$ is smooth, algebraic length 1 and (strongly) connected for all $n \geq 1$.

Theorem

If $\mathbb{G} = (G; E)$ is smooth, connected, algebraic length 1, and has Maltsev polymorphism, then it has join and meet polymorphisms.

The loop lemma

Lemma (Barto, Kozik, Niven; 2008)

If $\mathbb{G}$ is connected, smooth, algebraic length 1 and has a weak near-unanimity polymorphism, then $\mathbb{G}$ has a loop.

Theorem (Barto, Kozik, Niven; 2008)

The core of a smooth digraph with a weak near-unanimity polymorphism is a disjoint union of cycles.

Proposition

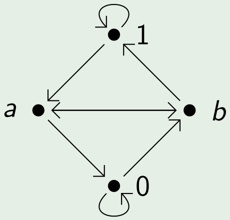
If $\mathbb{G}^{\mathbb{G}}$ is strongly connected, then $\mathbb{G}$ has a loop.

Take a path $\mathrm{id} \to f_1 \to f_2 \to \ldots f_n \to \mathrm{id}$ where f_k is a constant map. Then $\mathrm{id} \cdot f_1 \cdots f_{n-1} \cdot f_n \to f_1 \cdot f_2 \cdots f_n \cdot \mathrm{id}$, so we have a loop at $f_1 \cdots f_n$, which is a constant map.

Connectivity in End($\mathbb{G}$)

Example

The following digraph $\mathbb{G}$ has Maltsev, join and meet semilattice polymorphisms.



It has only four endomorphisms: id, 0, 1 and inversion, they are all isolated. However, id is connected to 0 in $\mathbb{G}^{\mathbb{G}}$:

$$\mathrm{id} = x \wedge 1 \to x \wedge a \to x \wedge 0 = 0.$$

Required subgraph of $\mathbb{G}^{\mathbb{G}}$

Definition

$\mathrm{Pol}_1(\mathbb{G})$ is the induced subgraph of $\mathbb{G}^{\mathbb{G}}$ on the set of **unary polynomials** of the algebra $\mathbf{G} = (G; \mathrm{Hom}(\mathbb{G}))$.

Proposition

- $\mathrm{Pol}_1(\mathbb{G}) \leq \mathbf{G}^G$ *is generated by the identity and the constant maps*
- $\mathbb{G}$ *is an induced subgraph of* $\mathrm{Pol}_1(\mathbb{G})$ *on the set of constant maps*
- $\mathrm{Pol}_1(\mathbb{G})$ *is smooth if and only if* $\mathbb{G}$ *is smooth*
- *If* $\mathbb{G}$ *is smooth, connected and algebraic length* 1*, then every component of* $\mathrm{Pol}_1(\mathbb{G})$ *has algebraic length* 1

Proof.

For a polynomial $p = t(x, a_1, \dots, a_n)$ we can find an oriented cycle in $\mathbb{G}^n$ of algebraic length 1 going through $(a_1, \dots, a_n)$. Then the polymorphism $t \in \mathrm{Hom}(\mathbb{G}^{n+1}, \mathbb{G}) = \mathrm{Hom}(\mathbb{G}^n, \mathbb{G}^{\mathbb{G}})$ maps this cycle to a cycle in $\mathrm{Pol}_1(\mathbb{G})$.

Proposition

If $\mathbb{G}$ is smooth, connected and algebraic length 1*, then the connectedness relation on* $\mathrm{Pol}_1(\mathbb{G})$ *is a congruence.*

Definition

Let $\mathbf{A}$ be an algebra. Two unary polynomials $p, q \in \mathrm{Pol}_1(\mathbf{A})$ are **twins**, if there exist a term t of arity $n+1$ and constants $\bar{a}, \bar{b} \in A^n$ such that

$$p = t(x, \bar{a}) \quad \text{and} \quad q = t(x, \bar{b}).$$

The transitive closure of twin polynomials is the **twin congruence** τ of the algebra $\mathrm{Pol}_1(\mathbf{A})$.

Corollary

If $\mathbb{G}$ is smooth, connected and algebraic length 1*, then the twin congruence blocks are connected.*

The component of the identity

Definition

A map $f \in \mathbb{G}^{\mathbb{G}}$ is **idempotent**, if $f^2 = f$; it is a **retraction**, if $f \to f$ and $f^2 = f$; and it is **proper**, if $f \neq \mathrm{id}$.

Lemma (M, Zádori; 2012)

If $\mathbb{G}$ is reflexive or symmetric and the component of the identity in $\mathrm{End}(\mathbb{G})$ *contains something other than* id, *then it contains a proper retraction.*

Theorem

If the smooth component of id *in $\mathbb{G}^{\mathbb{G}}$ (or in any submonoid) contains a non-permutation, then it contains a proper retraction.*

Corollary

If $\mathbb{G}$ is smooth and the component of id *contains a constant map, then the smooth part of $\mathbb{G}^{\mathbb{G}}$ is connected, $\mathbb{G}$ is connected and it contains a loop.*

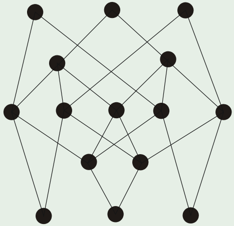
Required Maltsev condition

Example

The digraph $\mathbb{G} = (\{0, 1, 2\}; \neq)$ with 6 edges is connected, smooth, has algebraic length 1, and the identity in $\mathbb{G}^{\mathbb{G}}$ is isolated.

Example (Larose, Zádori; 2004)

This poset has a semilattice polymorhism, but not dismantable, so $\mathrm{Pol}_1(\mathbb{G}) = \mathrm{End}(\mathbb{G})$ is not connected.



How far can we push this?

- Smooth, algebraic length 1
- $\mathrm{Pol}_1(\mathbb{G})$
- Hobby-McKenzie terms (omitting types **1** and **5**)

Theorem

If $\mathbb{G}$ is a smooth, connected, algebraic length 1 with Hobby-McKenzie polymorphisms, then $\mathrm{Pol}_1(\mathbb{G})$ is connected (and $\tau = 1$).

- We prove that $\tau = 1$ for the twin congruence of $\mathrm{Pol}_1(\mathbb{G})$

Putting everything together

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- We prove that $\tau = 1$ for the twin congruence of $\mathrm{Pol}_1(\mathbb{G})$
- Clear for join semi-distributivite (omitting types **1**, **2** and **5**)
  - η_a is the projection kernel of $\mathrm{Pol}_1(\mathbb{G})$ onto its $a \in G$ coordinate
  - $\tau \vee \eta_a = 1$ because $p \,\eta_a\, p(a) \,\tau\, q(a) \,\eta_a\, q$,
  - use join semi-distributivity

$$\tau \vee \alpha = \tau \vee \beta \implies \tau \vee \alpha = \tau \vee (\alpha \wedge \beta)$$

to derive $\tau \vee (\bigwedge_a \eta_a) = 1$, that is $\tau = 1$.

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- Clear for join semi-distributivite (omitting types **1**, **2** and **5**)
- From Hobby-McKenzie TCT we get a ternary term m satisfying

$$m(\mathrm{id}, f, f) \;\tau\; m(f, f, \mathrm{id}) \;\tau\; \mathrm{id}\,.$$

  - $\mathrm{Pol}_1(\mathbb{G})$ is in a congruence join semi-distributive over modular variety
  - τ is solvable and $\mathrm{Pol}_1(\mathbb{G})/\tau$ is congruence permutable

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- So $m(x, a, b)$ and $m(a, b, x)$ are permutations for all $a, b \in G$

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Corollary

Every finite smooth connected digraph of algebraic length 1 with Hobby-McKenzie polymorphisms (omitting types **1** *and* **5***) has a loop.*

Corollary

A locally finite idempotent variety $\mathcal{V}$ has Hobby-McKenzie terms (omits types **1** *and* **5***) iff for every algebra* $\mathbf{A} \in \mathcal{V}$ *and connected subdirect relation* $\mathbf{E} \leq_{\mathrm{sd}} \mathbf{A}^2$ *of algebraic length 1 the graph* $\mathrm{Pol}_1((A; E))$ *is connected.*

Conjecture

If $\mathbb{G}$ is smooth, connected, algebraic length 1 and has Gumm polymorphisms, then it has a near-unanimity polymorphism.

Corollaries

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Connectivity in $\mathrm{Pol}_2^{\mathrm{id}}(\mathbb{G})$

Theorem (M, Zádori; 2012)

If $\mathbb{G}$ is reflexive, connected and has Gumm polymorphisms, then π_1 and π_2 are connected in the graph $\mathrm{Hom}^{\mathrm{id}}(\mathbb{G}^2, \mathbb{G})$ of idempotent binary morphisms.

Theorem

If $\mathbb{G}$ is a smooth, connected, algebraic length 1 digraph with Gumm polymorphisms, then the digraph $\mathrm{Pol}_2^{\mathrm{id}}(\mathbb{G})$ on the set of idempotent binary polynomials of $\mathbb{G}$ is connected (π_1 and π_2 are connected).

Proof.

Take a path $\mathrm{id} = f_0 \sim f_1 \sim \cdots \sim f_k = c$ in $\mathrm{Pol}_1(\mathbb{G})$ for some constant c.

$$d_i(x,x,y) = d_i(x, f_0(x), y) \sim d_i(x, f_1(x), y) \sim \cdots \sim d_i(x, f_k(x), y)$$
$$= d_i(x,c,y) = d_i(x, f_k(y), y) \sim \cdots \sim d_i(x, f_0(y), y) = d_i(x,y,y), \text{ and}$$
$$p(x,y,y) = p(f_0(x), f_0(y), y) \sim p(f_1(x), f_1(y), y) \sim \cdots \sim p(c,c,y) = y.$$

Idempotent subalgebras

Definition

An **idempotent subalgebra** of **A** is a subalgebra $\mathbf{B} \leq \mathbf{A}$ that is closed under all idempotent polynomials of **A**.

Proposition

If $\mathrm{Pol}_2^{\mathrm{id}}(\mathbb{G})$ *is connected, then every smooth idempotent subalgebra of* $\mathbb{G}$ *is connected.*

- Somewhat related to absorbing subalgebra (is it the same?)
- For Jónsson algebras $d_i(x, a, y)$ are idempotent binary polynomials for any choice of constant a.
- For Maltsev algebras $p(x, s(x), s(y))$ and $p(s(x), s(y), y)$ are idempotent binary polynomials for any choice of unary polynomial s.

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Problems

Let $\mathbb{G}$ be smooth connected digraph of algebraic length 1 with Taylor polymorphism

- $\mathrm{Pol}_1(\mathbb{G})/\tau$ is generated by two elements (id and c)
- Every τ block is smooth, connected, algebraic length 1
- Every τ block contains a loop (by the loop lemma)
- $\mathrm{Pol}_1(\mathbb{G})/\tau$ has a compatible semigroup operation (composition)
- Does $\mathrm{Pol}_1(\mathbb{G})/\tau$ have compatible semilattice (totally symmetric) operation?

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Let $\mathbf{A}$ be an algebra.

- If $\tau = 1$ in $\mathrm{Pol}_1(\mathbf{A})$, then the term condition $C(1, 1; \alpha)$ does not hold for any $\alpha < 1$. What are the connections between $\tau = 1$, term condition, rectangulation?
- If $\mathbf{A}$ has Willard-terms (omitting types **1** and **2**), does $\mathrm{Pol}_1(\mathbf{A})/\tau$ have a semilattice (totally symmetric) term?

Thank You!